

Hyperboloid preservation implies the Lorentz and Poincaré groups without dilations

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Abstract

An analogue of the Alexandrov-Zeeman theorem, based on hyperboloid preservation, as opposed to light cone preservation, is provided. This characterizes exactly the Poincaré group, as opposed to the Poincaré group extended by dilations. The hyperbolic analogue, as opposed to the cone-based Alexandrov-Zeeman theorem, is also valid in the case of a single space dimension. An orthochronous version holds as well, based on the preservation of forward hyperboloid shells.

Keywords: mass shell, momentum space, velocity hyperboloid, proper time velocity, Lorentz transformation, Poincaré transformation, orthochronous transformation, light cone, space-like, time-like, light-like, separation line, inertia line, optical line, Robb hyperplane, Minkowski space-time

1 Introduction and statement of Theorem 1

We propose to show that the Alexandrov-Zeeman (A-Z) theorem admits a tighter analogue, where the role played by light cones is taken by hyperboloids defined in $(n + 1)$ -dimensional space-time $\mathbb{R} \times \mathbb{R}^n$ by

$$\{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : (t - t_0)^2 - (\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 1\}$$

For $n = 3$, if the center (t_0, \mathbf{x}_0) of the hyperboloid is the origin $\mathbf{00} = 0000$, then the upper sheet of the hyperboloid (characterized by $t > 0$) represents the set of possible velocity four-vectors with respect to proper time, or the set of all forward unit tangent vectors of possible world-lines of particles

with positive mass. Alternatively, the upper sheet of the hyperboloid can be viewed as the mass shell consisting of the possible momentum four-vectors of unit mass particles.

The A-Z theorem can be formulated in several equivalent ways. The formulation we give in this section is based on light cones, as in [A-CJM], rather than on the causality relation as in [Z], but it refers to the doubly infinite (past and future) cone, as in Latzer's version [L]. Alexandrov's and Zeeman's earlier versions correspond to the orthochronous formulation given in Section 4. (See e.g. Goldblatt [G] for an overview.)

A *Lorentz transformation* of $(n+1)$ -dimensional space-time $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$ is any non-singular linear transformation of $\mathbb{R} \times \mathbb{R}^n$ mapping each vector $(t, \mathbf{x}) = (t, x_1, \dots, x_n)$ to a vector (s, \mathbf{y}) of the same Minkowski norm, *i.e.* such that $t^2 - \mathbf{x}^2 = t^2 - \mathbf{x} \cdot \mathbf{x} = t_1^2 - (x^2 + \dots + x_n^2)$ equals $s^2 - \mathbf{y}^2$. Lorentz transformations constitute the *Lorentz group* (sometimes called the extended Lorentz group as improper transformations and time reversals are not excluded). A *translation* is a transformation of $\mathbb{R} \times \mathbb{R}^n$ of the form $\mathbf{v} \mapsto \mathbf{v} + \mathbf{b}$, where $\mathbf{b} \in \mathbb{R} \times \mathbb{R}^n$ is some fixed vector. A *Poincaré transformation* is a Lorentz transformation followed by a translation, and these transformations constitute the *Poincaré group*. Finally, a *dilation* is a transformation of $\mathbb{R} \times \mathbb{R}^n$, necessarily linear and non-singular, of the form $\mathbf{v} \mapsto a\mathbf{v}$, where a is some fixed positive real number.

A-Z Theorem (Alexandrov [A1, A2, A-CJM], Zeeman [Z], Latzer [L])
With respect to the light cone

$$C = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : t^2 - \mathbf{x}^2 = 0\}$$

in $(n+1)$ -dimensional space-time, let G be the group of bijective transformations f of $\mathbb{R} \times \mathbb{R}^n$ satisfying

$$f[C + \mathbf{v}] = C + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$. If $n \geq 2$, then G is generated by the Poincaré group and the group of dilations.

The present paper is mainly devoted to deriving the following analogue:

Theorem 1 *Let $n \geq 1$. With respect to the hyperboloid*

$$H = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : t^2 - \mathbf{x}^2 = 1\}$$

in $(n + 1)$ -dimensional space-time, let G be the group of bijective transformations f of $\mathbb{R} \times \mathbb{R}^n$ satisfying

$$f[H + \mathbf{v}] = H + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$. Then G is exactly the Poincaré group.

In a slightly different formulation, the A-Z theorem says that the stabilizer G_0 of the origin in the group G (defined with reference to light cone preservation) is generated by the Lorentz group and the group of dilations. Theorem 1 says that the stabilizer G_0 (where G is now defined with reference to hyperboloid preservation) is precisely the Lorentz group. For recent references in connection with the Lorentz group and physical applications see e.g. [HL1, HL2, U, M, F].

Both the A-Z theorem and Theorem 1 will be shown to be consequences of Lemma 1 below (Sections 2 and 3). In Section 4 we show that the orthochronous version of Theorem 1, based on forward hyperboloid shells, also holds and characterizes the orthochronous Lorentz and Poincaré groups (Theorem 2 in Section 4).

The theorems will follow from a sequence of propositions of combinatorial nature about hyperboloids and the three types of lines in Minkowski space-time. Each one of these propositions can be verified with standard techniques of linear and convex geometry, using also appropriate Lorentz or Poincaré transformations. The detail of arguments will be indicated for some of the propositions. As restricting the discussion to $3 + 1$ dimensions offers no advantage in the present context, we carry out the argument for arbitrary dimension in full generality.

2 Collineations

Recall that in the vector space $\mathbb{R} \times \mathbb{R}^n$, a non-zero vector (t, \mathbf{x}) is classified as *space-like*, *light-like* or *time-like* according to whether $t^2 - \mathbf{x}^2$ is negative,

zero or positive. A line in $\mathbb{R} \times \mathbb{R}^n$, being a translate of a 1-dimensional subspace V , is classified as *space-like*, *light-like* or *time-like* according to whether the non-zero vectors in V are space-like, light-like or time-like. (These are also called *separation lines*, *optical lines* and *inertia lines*, respectively.) Two distinct points (vectors) $\mathbf{v}, \mathbf{w} \in \mathbb{R} \times \mathbb{R}^n$ are in *space-like*, *light-like* or *time-like relative position* according to whether the line $\{a\mathbf{v} + (1-a)\mathbf{w} : a \in \mathbb{R}\}$ through them is space-like, light-like or time-like.

A plane in $\mathbb{R} \times \mathbb{R}^n$, *i.e.* a translate of a 2-dimensional subspace, is a *Lorentz plane* if it contains two non-parallel light-like lines. Characterized alternatively, these are the planes that contain lines of all the three kinds (space-like, light-like and time-like). See Goldblatt [G] for a discussion of the different types of planes in space-time.

We shall make use of the facts that

(a) in a Lorentz plane P there are exactly two light-like lines L and L' through every point \mathbf{v} ,

(b) the Lorentz plane P through any two distinct intersecting light-like lines L and L' consists of the the union of all space-like lines intersecting $(L \cup L')$ in more than one point, plus the intersection point of L and L' .

The following lemma can itself be regarded as a version of the A-Z theorem, with stronger line preservation hypotheses but not depending on the assumption of more than one space dimension.

Lemma 1 *Let $n \geq 1$, and let G be the group of all bijective transformations f of $\mathbb{R} \times \mathbb{R}^n$ such that for every set of points $L \subseteq \mathbb{R} \times \mathbb{R}^n$*

(i) *L is a space-like line if and only if $f[L]$ is one,*

(ii) *L is a light-like line if and only if $f[L]$ is one.*

Then G is generated by the Poincaré group and the group of dilations.

Proof Clearly any $f \in G$ maps any pair $\{\mathbf{v}, \mathbf{w}\}$ of distinct points in space-like (respectively light-like) relative position to a point-pair $\{f(\mathbf{v}), f(\mathbf{w})\}$ in space-like (respectively light-like) position, and thus it must map point-pairs in time-like position to point-pairs in time-like position.

We claim that actually for any $f \in G$ and subset $L \subseteq \mathbb{R} \times \mathbb{R}^n$, L is a time-like line if and only if $f[L]$ is one. This, together with conditions (i) and

(ii) in the statement of the lemma, will imply that any $f \in G$ is an affine transformation. To prove the claim, let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R} \times \mathbb{R}^n$ be three distinct points such that $\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}$ and $\mathbf{u} - \mathbf{w}$ are space-like vectors, and let P be the plane containing $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then P is a Lorentz plane. Let L_1, L_2, L_3 be parallel light-like lines in P through $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively, and let $L'_1 \neq L_1, L'_2 \neq L_2, L'_3 \neq L_3$ be the other light lines in P through $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Let K_{12}, K_{13}, K_{23} be the lines such that

$$\begin{aligned} (L_1 \cap L'_2) \cup (L'_1 \cap L_2) &\subseteq K_{12} \\ (L_1 \cap L'_3) \cup (L'_1 \cap L_3) &\subseteq K_{13} \\ (L_2 \cap L'_3) \cup (L'_2 \cap L_3) &\subseteq K_{23} \end{aligned}$$

These lines are space-like. They are all parallel if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are collinear, and no two of them are parallel otherwise. Any $f \in G$ must map the Lorentz plane P to the Lorentz plane containing $f[L_1] \cup f[L'_1]$, and the points $f(\mathbf{u}), f(\mathbf{v}), f(\mathbf{w}) \in f[P]$ are collinear if and only if the lines $f[K_{12}], f[K_{13}], f[K_{23}]$ are all parallel, which proves the claim and shows that every $f \in G$ is an affine transformation.

Take now any $f \in G$. Let τ be the translation $\mathbf{v} \mapsto \mathbf{v} - f(\mathbf{00})$. The composition $\tau f \in G$ is a linear transformation. Let λ be a Lorentz transformation sending $\tau f(\mathbf{00})$ to a vector of the form $(a, \mathbf{0})$, $a \neq 0$, and let δ be the dilation $\mathbf{v} \mapsto a^{-1}\mathbf{v}$. Then $\delta\lambda\tau f \in G$ is a linear transformation fixing $(1, \mathbf{0})$ and it is easily seen to keep the hyperplane $S = \{(0, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ stable. It can be verified that each $(0, \mathbf{x}) \in S$ must be sent by $\delta\lambda\tau f$ to a vector $(0, \mathbf{y})$ satisfying $\|\mathbf{y}\| = \|\mathbf{x}\|$. Thus there is a Lorentz transformation ρ keeping S stable (space rotation) such that $\rho\delta\lambda\tau f$ is the identity, and $f = (\tau\lambda\delta\rho)^{-1}$. \square

The above lemma implies the A-Z theorem as follows. *Optical hyperplanes* (hyperplanes containing space-like and light-like lines but no time-like lines, also called *Robb hyperplanes* in [G]) are characterized in terms of light-like connection alone as sets of points in $\mathbb{R} \times \mathbb{R}^n$ of the form

$$L \cup \{\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n : \mathbf{v} - \mathbf{w} \text{ is not light-like for any } \mathbf{w} \in L\}$$

where L is any light-like line. Space-like lines are characterized as minimal non-empty non-singleton intersections of Robb hyperplanes, if $n \geq 2$. This fails for $n = 1$, but for $n \geq 2$, Lemma 1 can be applied to obtain the A-Z theorem.

3 Dilated hyperboloids and proof of Theorem 1

First we shall characterize point pairs in light-like relative position in terms of certain hyperboloids to which the two points belong (Lemma 2, its Corollary, and Proposition 1 below). This will allow the application of the A-Z theorem in the case of at least two space dimensions. In order to handle the case of a single space dimension, we shall go back to Lemma 1 in the previous section, but first we shall need to characterize also space-like lines in terms of hyperboloids (Propositions 2-5). These latter propositions will be stated and proved in full generality for any number $n \geq 1$ of space dimensions.

For any fixed vector (space-time point) $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$ and real number $r > 0$ consider the hyperboloid

$$H(\mathbf{v}, r) = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : t^2 - \mathbf{x}^2 = r^2\}$$

In particular, $H(\mathbf{0}, 1)$ is the hyperboloid H appearing in the statement of Theorem 1. For any set of points $S \subseteq \mathbb{R} \times \mathbb{R}^n$ and real number $m > 0$, consider the dilation of S by the factor m ,

$$mS = \{m\mathbf{u} : \mathbf{u} \in S\}$$

Note that for any positive real numbers m and r we have $mH(\mathbf{0}, r) = H(\mathbf{0}, mr)$.

Lemma 2 *Let $n \geq 1$. For any positive real number r we have*

- (i) $2H(\mathbf{0}, r) = \{\mathbf{v} : H(\mathbf{v}, r) \cap H(\mathbf{0}, r) \text{ is a singleton}\}$
- (ii) $H(\mathbf{0}, r) = \{\mathbf{v} : H(\mathbf{v}, r) \cap 2H(\mathbf{0}, r) \text{ is a singleton}\}$

Proof (i) To show that for every $\mathbf{v} \in 2H(\mathbf{0}, r)$ the intersection $H(\mathbf{v}, r) \cap H(\mathbf{0}, r)$ is a singleton, it is enough to see that this holds for $\mathbf{v} = (2r, \mathbf{0})$, because the Lorentz group acts transitively on $2H(\mathbf{0}, r)$, and any Lorentz transformation mapping \mathbf{u} to \mathbf{v} maps $H(\mathbf{u}, r)$ to $H(\mathbf{v}, r)$ and $H(\mathbf{0}, r)$ to itself. It can also be seen easily that for all $t \geq 0$, $t \neq 2r$, the intersection $H((t, 0 \dots 0), r) \cap H(\mathbf{0}, r)$ is either infinite or empty, and this, combined again with a transitivity consideration, shows that $H(\mathbf{v}, r) \cap H(\mathbf{0}, r)$ is a singleton only if $\mathbf{v} \in 2H(\mathbf{0}, r)$.

(ii) This is proved similarly. \square

Corollary *Let $n \geq 1$ and consider the hyperboloid $H = H(\mathbf{0}, 1)$. If a bijective transformation f of $\mathbb{R} \times \mathbb{R}^n$ satisfies*

$$f[H - \mathbf{v}] = H + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$, then it also satisfies

$$f[2^e H - \mathbf{v}] = 2^e H + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$ and all positive or negative integer exponents $e \in \mathbb{Z}$. \square

Proposition 1 *Two distinct points $\mathbf{u}, \mathbf{w} \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, are in light-like relative position if and only if for each integer $e \in \mathbb{Z}$ and $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$, at most one of \mathbf{u}, \mathbf{w} lies on the hyperboloid $H(\mathbf{v}, 2^e)$.*

Proof First, no two distinct points in light-like relative position lie on any hyperboloid $H(\mathbf{v}, r)$.

Second, for every positive real x there is a positive real t such that both $(t, -x, 0, \dots, 0)$ and $(t, x, 0, \dots, 0)$ lie on $H(\mathbf{0}, 1)$. By translation and Lorentz transformation, any two points in space-like relative position lie on some $H(\mathbf{v}, 1)$.

Third, let $t > 0$. Take any integer e such that $2^e \leq t$. Then there is a positive real x such that both $(t, -x, 0, \dots, 0)$ and $(t, x, 0, \dots, 0)$ lie on $2^e H(\mathbf{0}, 1) = H(\mathbf{0}, 2^e)$. Again by translation and Lorentz transformation, any two points in time-like relative position lie on some $H(\mathbf{0}, 2^e)$ for an appropriate e . \square

It is now clear that a transformation f belonging to the hyperboloid preserving group G of Theorem 1 satisfies for all points \mathbf{v} the light cone preservation condition $f[C + \mathbf{v}] = C + f(\mathbf{v})$ appearing in the A-Z theorem. Also, hyperboloid preservation rules out non-trivial dilations. This proves Theorem 1 for all $n \geq 2$. In order to include also the case $n = 1$ we shall characterize, in terms of hyperboloids, first point-pairs in space-like position, then space-like collinearity. An argument along the second part of the proof of Proposition 1 shows the following:

Proposition 2 *Two distinct points $\mathbf{u}, \mathbf{w} \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, are in space-like relative position if and only if for all integers $z \in \mathbb{Z}$ and $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$, there is a $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$, such that both \mathbf{u}, \mathbf{w} lie on the hyperboloid $H(\mathbf{v}, 2^z)$. \square*

Each hyperboloid $H(\mathbf{v}, r)$ has two topologically connected components, called *shells*. A hyperboloid shell C is *forward* or *backward* according to whether the time projection $\{t \in \mathbb{R} : (t, \mathbf{x}) \in C\}$ is bounded from below or from above. Two shells have the *same orientation* if both are forward or both are backward, otherwise they have *opposite orientation*. For each shell C there is only one point \mathbf{v} and only one positive constant r such that C is one of the two shells of opposite orientation making up the hyperboloid $H(\mathbf{v}, r)$: the constant r and the point \mathbf{v} are the *radius* and the *center* of the shell C . If the radius r is an integer power of 2, *i.e.* if $r = 2^e$ for some $e \in \mathbb{Z}$, then we shall call C a *standard shell*.

Proposition 3 *Two distinct points $\mathbf{u}, \mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, lying on a hyperboloid $H(\mathbf{v}, r)$, lie on the same connected component of this hyperboloid if and only if they are in space-like relative position. \square*

Proposition 4 *In $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, let C be a hyperboloid shell with center \mathbf{v} and K a shell with center \mathbf{w} . Assume that the centers \mathbf{v} and \mathbf{w} are in space-like relative position. Then $C \cap K = \emptyset$ if and only if C and K have opposite orientation. \square*

A point $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$ is *between* points \mathbf{u}, \mathbf{w} if $\mathbf{v} = a\mathbf{u} + (1 - a)\mathbf{w}$ for some $0 \leq a \leq 1$. This implies that the three points are collinear. Conversely, of any three collinear points one is always between the other two.

Proposition 5 *Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct points in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, each two of them being in space-like relative position. Then \mathbf{v} is between \mathbf{u} and \mathbf{w} if and only if the following holds:*

Whenever \mathbf{u}, \mathbf{w} lie on some standard hyperboloid shell C and \mathbf{v} lies on some standard shell K , these respective shells C and K are of the same orientation or they possess a common point.

Proof Suppose \mathbf{v} is between \mathbf{u} and \mathbf{w} but there are disjoint hyperboloid shells C and K of opposite orientation with $\mathbf{u}, \mathbf{w} \in C$ and $\mathbf{v} \in K$. By an appropriate Poincaré transformation we can map $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to points $\mathbf{u}', \mathbf{v}', \mathbf{w}'$ such that

$$\mathbf{u}' = (t, -x, 0, \dots, 0) \quad \mathbf{w}' = (t, x, 0, \dots, 0)$$

for some positive real numbers t and x . Then $\mathbf{v}' = (t, q, 0, \dots, 0)$ with $-x < q < x$. The Poincaré transformation used maps C and K to disjoint hyperboloid shells C' and K' which are also of orientation opposite to

each other. But $\mathbf{u}', \mathbf{w}' \in C'$ and $\mathbf{v}' \in K'$, and the shells C' and K' must have a common point in the plane $\{(t, x_1, 0, \dots, 0) : t, x_1 \in \mathbb{R}\}$, a contradiction proving the "common orientation or common point" condition.

Suppose that \mathbf{v} is not between \mathbf{u} and \mathbf{w} . By an appropriate Poincaré transformation f we can map $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to points $\mathbf{u}', \mathbf{v}', \mathbf{w}'$ of the form

$$\mathbf{u}' = (t, -x, 0, \dots, 0) \quad \mathbf{w}' = (t, x, 0, \dots, 0) \quad t, x > 0 \quad \mathbf{v}' = (x_0, x_1, \dots, x_n)$$

and such that either

$$x_0 = t \quad x_1 \notin [-x, x] \quad x_2 = \dots = x_n = 0$$

or $x_0 \neq t$. Clearly in both cases there exist two disjoint standard hyperboloid shells C' and K' of opposite orientation such that $\mathbf{u}', \mathbf{w}' \in C'$ and $\mathbf{v}' \in K'$. Then the inverse image shells $C = f^{-1}[C']$ and $K = f^{-1}[K']$ containing \mathbf{u}, \mathbf{v} and \mathbf{w} , respectively, are also disjoint and of opposite orientation. \square

Suppose that a transformation f belongs to the hyperboloid preserving group G of Theorem 1, and that the number n of space dimensions is any positive integer, possibly equal to 1. By Propositions 2 and 3, and the Corollary of Lemma 2, f must map standard hyperboloid shells to standard shells. Using Propositions 4 and 5 it can be concluded that f maps space-like lines to space-like lines. Theorem 1 now follows from Lemma 1.

4 Time orientation

A Lorentz transformation is *orthochronous* (preserves time orientation) if it maps the vector $(1, \mathbf{0})$ to a vector (t, \mathbf{x}) with $t > 0$. These transformations constitute the *orthochronous Lorentz group*. Together with all translations this group generates the *orthochronous Poincaré group*, consisting of *orthochronous Poincaré transformations*. Clearly all of these preserve forward light cones and forward hyperboloids. The following corresponds to Alexandrov's and Zeemans's original formulations of A-Z:

A-Z Theorem, Orthochronous Formulation (Alexandrov[A1, A2, A-CJM], Zeeman [Z]) *With respect to the forward light cone*

$$C^+ = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : t^2 - \mathbf{x}^2 = 0, t \geq 0\}$$

in $(n + 1)$ -dimensional space-time, let G be the group of bijective transformations f of $\mathbb{R} \times \mathbb{R}^n$ satisfying

$$f[C^+ + \mathbf{v}] = C^+ + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$. If $n \geq 2$, then G is generated by the orthochronous Poincaré group and the group of dilations.

(The deduction from the version stated in Section 1 is easy, as a transformation f preserving forward light cones necessarily preserves light-like lines and thus preserves light cones. Any such f is then a dilation followed by a Poincaré transformation. If the dilation factor is a , then the Lorentz transformation $\mathbf{v} \mapsto a^{-1}[f(\mathbf{v}) - f(0, \mathbf{0})]$ is obviously orthochronous.)

Considering hyperboloid shells instead of cones, we obtain the orthochronous version of Theorem 1:

Theorem 2 *Let $n \geq 1$. With respect to the forward hyperboloid shell*

$$H^+ = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : t^2 - \mathbf{x}^2 = 1, t > 0\}$$

in $(n + 1)$ -dimensional space-time, let G be the group of bijective transformations f of $\mathbb{R} \times \mathbb{R}^n$ satisfying

$$f[H^+ + \mathbf{v}] = H^+ + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$. Then G is exactly the orthochronous Poincaré group.

Proof The backward shell $H^- = H(\mathbf{0}, 1) \setminus H^+ = -H^+$ also satisfies

$$f[H^- + \mathbf{v}] = H^- + f(\mathbf{v})$$

and therefore

$$f[H(\mathbf{0}, 1) + \mathbf{v}] = H(\mathbf{0}, 1) + f(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R} \times \mathbb{R}^n$ and all $f \in G$. By Theorem 1, G is a subgroup of the Poincaré group, and it obviously contains all orthochronous Poincaré transformations. But it can only contain orthochronous transformations, because for every $f \in G$ the Lorentz transformation $\mathbf{v} \mapsto f(\mathbf{v}) - f(0, \mathbf{0})$ is orthochronous. \square

5 Concluding remarks

Theorems 1 and 2 essentially say that for space-time transformations, not assumed a priori to be linear or continuous, preservation of hyperboloids (respectively of forward hyperboloid shells) implies light cone preservation (respectively forward light cone preservation, i.e. preservation of causality). The discussion was carried out for $\mathbb{R} \times \mathbb{R}^n$ without restriction on n , and without emphasizing the particular significance of the case $n = 3$. Copies of all lower dimensional space-time models are embedded in all higher dimensional models. The case $n = 1$ exhibits some specifics which can be dealt with either by restricting attention to $n \geq 2$ as in the A-Z theorem, or by going around this result as we have done in Section 3 to establish Theorem 1 for all $n \geq 1$. We note that the specifics of the case $n = 1$ allow the consideration of superluminal $(1 + 1)$ -dimensional frames of reference, when viewed in themselves and not as embedded in $(2 + 1)$ or $(3 + 1)$ dimensional space-time (see Parker [P]).

Theorem 1 is formulated with reference to the hyperboloid $H = H(\mathbf{0}, 1)$ of radius 1 whose upper component constitutes the unit mass shell H^+ that is shown by Theorem 2 to be the basic geometric invariant of the orthochronous Lorentz group. It is straightforward to see that these theorems also hold if reformulated with reference to any of the hyperboloids $H = H(\mathbf{0}, m)$, $m > 0$.

References

- [A1] A.D. Alexandrov, On Lorentz transformations, Sessions Math. Seminar, Leningrad Section of the Mathematical Institute, 15 September 1949 (abstract, in Russian)
- [AO] A.D. Alexandrov, V.V. Ovchinnikova, Note on the foundations of relativity theory, Vestnik Leningrad Univ. 11 (1953) 95-100 (in Russian)
- [A-CJM] A.D. Alexandrov, A contribution to chronogeometry, Canadian J. Math. 19 (1967) 1119-1128
- [F] S. Foldes, The Lorentz group and its finite field analogs: local isomorphism and approximation, J. Mathematical Physics 49 (9) (2008) 093512:1-10.

- [G] R. Goldblatt, Orthogonality and Spacetime Geometry, Springer 1987
- [HL1] H.-K. Hong, C.-S. Liu, Lorentz group on Minkowski spacetime for construction of the two basic principles of plasticity, *Int. J. Non-Linear Mechanics* 36 (2001) 679-686
- [HL2] H.-K. Hong, C.-S. Liu, Some physical models with Minkowski spacetime structure and Lorentz group symmetry, *Int. J. Non-Linear Mechanics* 36 (2001) 1075-1084
- [L] R.W. Latzer, Non-directed light signals and the structure of time, *in Space, Time and Geometry*, P. Suppes (ed.) D. Reidel 1973, pp. 321-365
- [M] V. Moretti, The interplay of the polar decomposition theorem and the Lorentz group, *Lecture Notes, Seminario Interdisciplinare di Matematica* 5-153 (2006) 18 pages, also ArXiv:math-ph/0211047v1 at www.arxiv.org
- [P] L. Parker, Faster-than-light inertial frames and tachyons, *Physical Review* 188 No. 5 (1969) 2287-2292
- [U] H.K. Urbantke, Lorentz transformations from reflections: some applications, *Found. Phys. Lett.* 16 (2003) 111-117
- [Z] E.C. Zeeman, Causality implies the Lorentz group, *J. Mathematical Physics* 5 (1964) 490-493